Analize III, pismeni ispit, 16.06.2014.

1. Ispitati neprekidnost funkcije $f(x,y) = \begin{cases} \frac{(x-1)^2 \ln x}{(x-1)^2+y^2}, & (x,y) \neq (1,0) \\ 0, & (x,y) = (1,0) \end{cases}$.

2. Primjenom dvostrukog integrala izračunati površinu figure ograničene linijom $(x^2+y^2)^3 = a^2(x^4+y^4)$.

3. Izračunati pomoću Stoksove formule (ili direktno)

$$I = \oint_c y^2 \mathrm{d}x + z^2 \mathrm{d}y + x^2 \mathrm{d}z$$

ako je c kontura trougla $\triangle ABC$, $A(2,0,0)$, $B(0,1,0)$, $C(0,0,-3)$, pređena u pozitivnom smislu.

4. Izračunati integral $I = \iint_S z^3 \mathrm{d}x\mathrm{d}y + x^3 \mathrm{d}y\mathrm{d}z + y^3 \mathrm{d}z\mathrm{d}x$ gdje je S-vanjska strana konusne površi $G : x^2+y^2 \leq z^2$, $0 \leq z \leq 1$.

VAŽNO: Ovaj papir treba predati zajedno s rješenjima zadataka! Prije rješenja prepisati postavku (tekst) zadatka. Ispit pisati isključivo hemijskom olovkom plave ili crne tinte.

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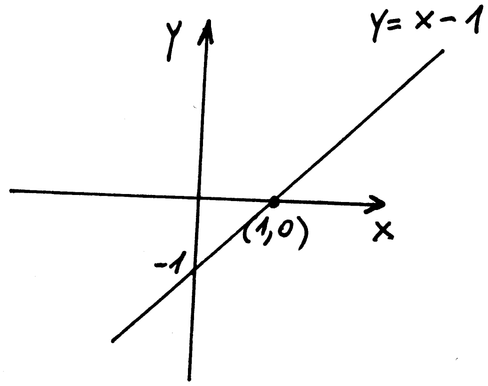
Zadaci su skinuti sa stranice ff.unze.ba/nabokov.
Za uočene greške pisati na *infoarrt@gmail.com*

Ispitati neprekidnost f-je $f(x,y) = \begin{cases} \dfrac{(x-1)^2 \ln x}{(x-1)^2+y^2}, & (x,y)\neq(1,0) \\ 0, & (x,y)=(1,0) \end{cases}$.

Rj. Jedina sumnjiva tačka u kojoj f-ja može imati prekid je tačka $(1,0)$. F-ja će biti neprekidna u ovoj tački akko

$$\lim_{(x,y)\to(1,0)} f(x,y) = f(1,0)$$



tj. akko $\displaystyle\lim_{(x,y)\to(1,0)} \frac{(x-1)^2 \ln x}{(x-1)^2+y^2} = 0$.

Posmatrajmo približavanje tački $(1,0)$ preko prave $y=0$.

$$\lim_{(x,0)\to(1,0)} \frac{(x-1)^2 \ln x}{(x-1)^2+0^2} = \lim_{(x,0)\to(1,0)} \ln x = 0$$

Posmatrajmo približavanje tački $(1,0)$ preko prave $y=x-1$

$$\lim_{(x,x-1)\to(1,0)} \frac{(x-1)^2 \ln x}{(x-1)^2+(x-1)^2} = \lim_{(x,x-1)\to(1,0)} \frac{\ln x}{2} = 0$$

Odavde možemo naslutiti da je možda vrijednost ovog limesa u tački $(1,0)$ jednaka 0.

Prisjetimo se teoreme "dva policajca":

$$\forall (x,y)\in\mathbb{R}^2 \quad g(x,y)\leq f(x,y)\leq h(x,y) \ \text{ i } \lim_{(x,y)\to(a,b)} g(x,y) = \lim_{(x,y)\to(a,b)} h(x,y) = M$$

$$\Longrightarrow \lim_{(x,y)\to(a,b)} f(x,y) = M \iff |f(x,y)-M|\to 0,\ (x,y)\to(a,b)$$

$(x+1)^2 \geq 0 \quad \forall x\in\mathbb{R}$

$(x-1)^2+y^2 \geq 0 \quad \forall x,y\in\mathbb{R}$

$$(x-1)^2+y^2 \geq (x+1)^2 \Rightarrow 0 \leq \frac{(x-1)^2}{(x-1)^2+y^2} \leq 1 \Rightarrow$$

$$\Rightarrow 0 \leq \frac{(x-1)|\ln x|}{(x-1)^2+y^2} \leq |\ln x| \to 0,\ (x,y)\to(1,0) \overset{\text{Teor. dva polic.}}{\Longrightarrow} \lim_{(x,y)\to(1,0)} \frac{(x-1)^2 \ln x}{(x-1)^2+y^2} = 0$$

F-ja je neprekidna.

Primjenom dvostrukog integrala izračunati površinu figure ograničene linijom $(x^2+y^2)^3 = a^2(x^4+y^4)$.

Rj.

Izgled date linije ne igra nikakvu ulogu u rješavanju ovog zadatka.

$$P=\iint_D dx\,dy$$

Uvedimo polarne koordinate

$$x=\rho\cos\varphi$$

$$y=\rho\sin\varphi$$

$$dx\,dy=\rho\,d\rho\,d\varphi$$

$$x^2+y^2=\rho^2$$

$$(x^2+y^2)^3=a^2(x^4+y^4)$$

$$(\rho^2)^3=a^2(\rho^4\cos^4\varphi+\rho^4\sin^4\varphi)$$

$$\rho^6=a^2\rho^4(\cos^4\varphi+\sin^4\varphi) \quad /:\rho^4$$

$$\rho^2=a^2(\cos^4\varphi+\sin^4\varphi)$$

Primjetimo da je data jednakost definisana za $\forall\varphi\in(0,2\pi)$

$$D \xrightarrow{\text{transformacije}} D': \begin{cases} 0\le\varphi\le 2\pi \\ 0\le\rho\le\sqrt{a^2(\cos^4\varphi+\sin^4\varphi)} \end{cases}$$

$$P=\iint_{D'}\rho\,d\rho\,d\varphi=\int_0^{2\pi}d\varphi\int_0^{\sqrt{a^2(\cos^4\varphi+\sin^4\varphi)}}\rho\,d\rho=\frac{1}{2}\int_0^{2\pi}a^2(\cos^4\varphi+\sin^4\varphi)\,d\varphi$$

$$\int_0^{2\pi}\cos^4\varphi\,d\varphi=\ldots=\frac{3\pi}{4} \qquad \int_0^{2\pi}\sin^4\varphi\,d\varphi=\ldots=\frac{3\pi}{4}$$

$$P=\frac{a^2}{2}\cdot\frac{\cancel{6}^{3}\pi}{\cancel{4}_{2}}=\frac{3a^2}{4}\pi$$ tražena površina

(#) Izračunati pomoću Stoksove formule (ili direktno)

$$I=\oint\limits_{c} y^2dx+z^2dy+x^2dz,$$ ako je c kontura trougla ΔABC,

$A(2,0,0)$, $B(0,1,0)$, $C(0,0,-3)$, pređena u pozitivnom smislu.

Rj. I način

Prisjetimo se

$$\int\limits_{c} P_{(x,y,z)}\,dx+Q_{(x,y,z)}\,dy+R_{(x,y,z)}\,dz=\iint\limits_{S}\begin{vmatrix} dydz & dxdz & dxdy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

površinski integral druge vrste

$$\frac{\partial Q}{\partial x}=0,\quad \frac{\partial P}{\partial y}=2y \qquad \frac{\partial R}{\partial x}=2x,\quad \frac{\partial P}{\partial z}=0$$

$$\frac{\partial R}{\partial y}=0,\quad \frac{\partial Q}{\partial z}=2z$$

$$I=\oint\limits_{c} y^2dx+z^2dy+x^2dz=\left|\begin{matrix}\text{Formula}\\ \text{Stoksa}\end{matrix}\right|=\iint\limits_{S}-2z\,dydz-2x\,dxdz-2y\,dxdy$$

gdje S predstavlja dio površi u prostoru ograničen trouglom ΔABC.

$A(2,0,0)$
$B(0,1,0)$
$C(0,0,-3)$

Jednačina ravni kroz tri tačke

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0$$

$$\begin{vmatrix} x-2 & y & z \\ -2 & 1 & 0 \\ -2 & 0 & -3 \end{vmatrix}=0$$

koji je najlakši način da zapamtimo ovu formulu?

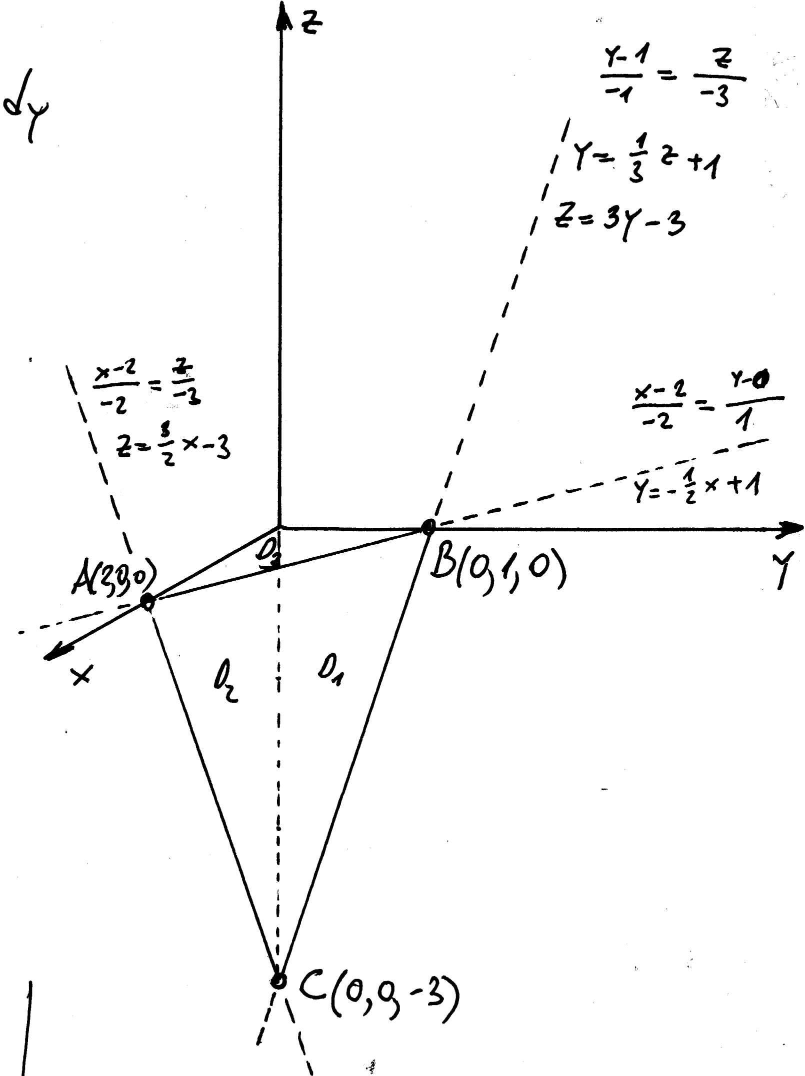
$$-3(x-2)-6y+2z=0$$

$$-3x-6y+2z+6=0$$

$$\vec{n}=(-3,-6,2)$$

$$|\vec{n}|=\sqrt{9+36+4}=7 \qquad \vec{n}_0=\left(-\frac{3}{7},-\frac{6}{7},\frac{2}{7}\right)$$

$$I = (-2)\iint\limits_S z\,dy\,dz + x\,dx\,dz + y\,dx\,dy$$



$$I_1 = \iint\limits_S z\,dy\,dz = \left| \begin{array}{l} \cos\alpha < 0 \\ D_1: \begin{cases} 0 \le y \le 1 \\ 3y-3 \le z \le 0 \end{cases} \end{array} \right| =$$

$$= -\iint\limits_D z\,dy\,dz = -\int_0^1 dy \int_{3y-3}^{0} z\,dz =$$

$$= \dots = +\frac{3}{2}$$

$$I_2 = \iint\limits_S x\,dx\,dz = \left| \begin{array}{l} \cos\beta < 0 \\ D_2: \begin{cases} 0 \le x \le 2 \\ \frac{3}{2}x - 3 \le z \le 0 \end{cases} \end{array} \right|$$

$$= -\int_0^2 x\,dx \int_{\frac{3}{2}x-3}^{0} dz = \dots = -2$$

$$I_3 = \iint\limits_S y\,dx\,dy = \left| \begin{array}{l} \cos\gamma > 0 \\ D_3: \begin{cases} 0 \le x \le 2 \\ 0 \le y \le -\frac{1}{2}x + 1 \end{cases} \end{array} \right| = +\int_0^2 dx \int_0^{-\frac{1}{2}x+1} y\,dy = \dots = \frac{1}{3}$$

Prema tome $I = (-2)\left(\frac{3}{2} - 2 + \frac{1}{3}\right) = \frac{1}{3}$ tražena vrijednost

II način

Koristimo formulu

$$\int\limits_C P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz = \iint\limits_S \begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} dS$$

$$I=\oint_C y^2dx+z^2dy+x^2dz=\left|\begin{matrix}\text{formula}\\ \text{stoksa}\end{matrix}\right|=$$

$$=\iint_S(-2z\cos\alpha-2x\cos\beta-2y\cos\gamma)\,dS=\left|\begin{matrix}\cos\alpha=-\frac{3}{7}\\ \cos\beta=-\frac{6}{7}\\ \cos\gamma=\frac{2}{7}\end{matrix}\right|$$

$$=\iint_S\left(\frac{6}{7}z+\frac{12}{7}x-\frac{4}{7}y\right)dS=$$

ravan kroz tačke A,B,C

$$-3x-6y+2z+6=0$$

$$2z=3x+6y-6$$

$$z=\frac{3}{2}x+3y-3$$

označimo sa D projekciju oblasti S na xOy ravan

$$dS=\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}$$

$$\frac{\partial z}{\partial x}=\frac{3}{2}\qquad dS=\sqrt{\frac{49}{4}}=\frac{7}{2}$$

$$\frac{\partial z}{\partial y}=3$$

$$=\frac{7}{2}\iint_D\left(\frac{6}{7}\left(\frac{3}{2}x+3y-3\right)+\frac{12}{7}x-\frac{4}{7}y\right)dx\,dy$$

$$=\frac{7}{2}\iint_D\left(3x+2y-\frac{18}{7}\right)dx\,dy=\frac{7}{2}\int_0^2dx\int_0^{-\frac{1}{2}x+1}\left(3x+2y-\frac{18}{7}\right)dy=\frac{7}{2}\cdot\frac{2}{21}=\frac{1}{3}$$

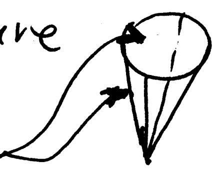
tražena vrijednost

(#) Izračunati integral

$$I=\iint\limits_S x^3\,dy\,dz+y^3\,dz\,dx+z^3\,dx\,dy$$

gdje je S - vanjska strana konusne površi $G: x^2+y^2\le z^2$, $0\le z\le 1$.

Rj.-upute:

Označimo sa I_1 integral po površini S_1 čitave date figure 

S_1

(omotača i baze) a sa I_2 integral po gornjoj strani baze S_2. Tada je $I=I_1-I_2$. U integralu I_1 primjenom formule Gauss-Ostrogradskog

$$I_1=3\iiint\limits_\Omega (x^2+y^2+z^2)\,dx\,dy\,dz$$

Prelazeći na cilindrične koordinate dobićemo sljedeće

$$I_1=3\int_0^1 dz\int_0^{2\pi}d\varphi\int_0^z(\rho^2+z^2)\rho\,d\rho=\ldots=\frac{9}{10}\pi$$

Računajući integral po osnovici konusa

$$I_2=\iint\limits_{S_2} x^3\,dy\,dz+y^3\,dz\,dx+z^3\,dx\,dy=\iint\limits_{S_2}dx\,dy=\pi$$

Prema tome $I=-\dfrac{\pi}{10}$ tražena vrijednost.